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LETTER TO THE EDITOR

Markov processes and a multiple generating function of product of generalized Laguerre polynomials

Poh-Aun Lee
Department of Mathematics, University of Malaya, 50603 Kuala Lumpur, Malaysia

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Abstract. From the spectral representation of the transition probability of birth-and-death processes, Karlin and McGregor show that the transition probability for the infinite server Markovian queue is in the form of a diagonal sum involving a product of Charlier polynomials. By using Meixner's bilinear generating formula for the Charlier polynomials and the Markov property, a multiple generating for the Charlier polynomials is deduced from the Chapman–Kolmogorov equation. The resulting formula possesses the same genre of a multiple generating function for the generalized Laguerre polynomials discussed by Messina and Paladimo, the explicit solution of which is recently given by the present author.

1. Introduction

In a previous communication [1], we discussed the explicit evaluation of a multiple generating function involving the product of generalized Laguerre polynomials

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\ {}_1F_1(-n;\alpha+1;x)$$

of the form

$$I_n = \sum_{s_1,s_2,\ldots,s_n=0}^{\infty} z_1^{s_1} z_2^{s_2}\cdots z_n^{s_n} L_{s_2}^{(s_1-s_2)}(x) L_{s_3}^{(s_2-s_3)}(x)\cdots L_{s_1}^{(s_n-s_1)}(x). \tag{1}$$

It can be shown that I_n is a symmetric function of the variables $z_1, z_2, \ldots, z_n$, and is explicitly given by

$$I_n = \frac{1}{1-\sigma_n}\exp - \left\{\frac{x}{1-\sigma_n}\Big[(n-1)\sigma_n + \psi(1) - 1\Big]\right\} \qquad n \geqslant 1 \tag{2}$$

where the elementary symmetric functions are defined as

$$\sigma_0 = 1$$

$$\sigma_1 = \sum_{i=1}^{n} z_i$$

$$\sigma_2 = \sum_{i<j}^{n} z_i z_j$$

$$\sigma_3 = \sum_{i<j<k}^{n} z_i z_j z_k$$

$$\vdots$$

$$\sigma_n = z_1 z_2 \cdots z_n.$$

and $\psi(t)$, the generating function of σ_i, $0 \leqslant i \leqslant n$, is given by

$$\psi(t) = \sum_{i=0}^{n} t^i \sigma_i = \prod_{i=1}^{n} (1 + z_i t).$$

The multiple generating function in (1) was originally discussed by Messina and Paladimo [2], who gave a solution, in implicit form, by appealing to properties of the trace of a certain sequence of operators. Apart from its relevance in the theory of special functions, this generating function is also encountered in a variety of situations involving researches in physics and chemistry. (See [2] for further references.)

We will now show that an allied sum of the form of (1) is intimately connected with the transition probability of a certain stochastic process arising from the infinite server Poisson queueing system. The Markovian nature of the process, and in particular, the Chapman–Kolmogorov equation, allows us to deduce an explicit expression of a sum which bears a close resemblance to the solution given in (2). The method makes use of the result of Karlin and McGregor [3, 4] on the spectral representation of the transition probability of the birth-and-death process in terms of orthogonal polynomials, the outline of which is briefly described below.

2. The birth-and-death process

A birth-and-death process is a stationary Markov process $\{X(t), t \geqslant 0\}$ whose state space S is the set of non-negative integers $\{0, 1, 2, \ldots\}$, and whose transition probabilities

$$P_{ij}(t) = \Pr[X(t) = j | X(0) = i] \qquad i, j \in S \quad t \geqslant 0$$

satisfy the following infinitesimal transition scheme as $\Delta t \to 0$

$$P_{ij}(\Delta t) = \begin{cases} \lambda_i \Delta t + \mathrm{o}(\Delta t) & \text{if } j = i + 1 \\ \mu_i \Delta t + \mathrm{o}(\Delta t) & \text{if } j = i - 1 \\ 1 - (\lambda_i + \mu_i)\Delta t + \mathrm{o}(\Delta t) & \text{if } j = i \\ \mathrm{o}(\Delta t) & \text{if } | j - i | > 1 \end{cases}$$

where $\lambda_i > 0$ for $i \geqslant 0$, $\mu_i > 0$ for $i \geqslant 1$ and $\mu_0 \geqslant 0$.

The infinitesimal matrix of the birth-and-death process is of the form

$$\mathbf{A} = \begin{pmatrix} -(\lambda_0 + \mu_0) & \lambda_0 & 0 & \cdots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & \cdots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

This matrix determines a system of orthogonal polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ by means of the following three-term recurrence relation

$$-x Q_n(x) = \mu_n Q_{n-1}(x) - (\lambda_n + \mu_n) Q_n(x) + \lambda_n Q_{n+1}(x) \qquad n \geqslant 0 \qquad (3)$$

where $Q_n(x) = 0$ for $n < 0$, and $Q_0(x) = 1$.

It is shown in [3] that there is a positive measure $\phi(x)$ on $0 \leqslant x < \infty$ for which the following orthogonality conditions hold

$$\int_0^\infty Q_i(x) Q_j(x)\, \mathrm{d}\phi(x) = \frac{\delta_{ij}}{\pi_j} \qquad i, j \in S$$

where $\pi_0 = 1, \quad \pi_n = \lambda_0 \lambda_1 \cdots \lambda_{n-1} / \mu_1 \mu_2 \cdots \mu_n$.

Moreover, the transition probability matrix $\mathbf{P}(t) = (P_{ij}(t))$ of the process which is uniquely determined by $\mathbf{A}$, has the following explicit spectral representation

$$P_{ij}(t) = \pi_j \int_0^\infty \mathrm{e}^{-xt} Q_i(x) Q_j(x)\, \mathrm{d}\phi(x) \qquad i, j \in S. \tag{4}$$

In addition, from the Markovian nature of the process, $\mathbf{P}(t)$ will satisfy the Chapman–Kolmogorov equation

$$\mathbf{P}(s+t) = \mathbf{P}(s)\mathbf{P}(t) \qquad s, t \geqslant 0$$

or equivalently

$$P_{ij}(s+t) = \sum_{k=0}^{\infty} P_{ik}(s) P_{kj}(t) \qquad i, j \in S. \tag{5}$$

3. The M/M/∞ queueing process

For the infinite server queue M/M/∞ with Poisson input and exponential service, we have

$$\lambda_n = \lambda \qquad n \geqslant 0$$

$$\mu_n = n\mu \qquad n \geqslant 1.$$

From (3), the orthogonal polynomials associated with this process, denoted by $Q_n(x) \equiv Q_n(x, \lambda, \mu)$, satisfy the following three-term recurrence relation:

$$-x Q_n(x) = n\mu Q_{n-1}(x) - (\lambda + n\mu) Q_n(x) + \lambda Q_{n+1}(x) \qquad n \geqslant 0$$

where $Q_n(x) = 0$ for $n < 0$, and $Q_0(x) = 1$.

As shown in [5], this is identical with the Charlier polynomials $c_n(x; a)$ which satisfy the following three-term recurrence relation [5]:

$$-x c_n(x; a) = n c_{n-1}(x; a) - (n + a) c_n(x; a) + a c_{n+1}(x; a) \qquad n \geqslant 0$$

where $c_0(x; a) = 1,\ c_n(x; a) = 0$ for $n < 0$. Thus on identification we have

$$Q_n(x) = c_n\left(\frac{x}{\mu}; \frac{\lambda}{\mu}\right).$$

It is well known [6] that the Charlier polynomials are a system of discrete orthogonal polynomials with respect to the following Poisson probability distribution:

$$p(x) = \mathrm{e}^{-a} \frac{a^x}{x!} \qquad x = 0, 1, 2, \ldots.$$

Hence, the spectral measure $\phi(x)$ consists of masses

$$\mathrm{d}\phi(x) = \mathrm{e}^{-a} \frac{a^n}{n!} \qquad \text{at } x_n = n\mu \quad n = 0, 1, 2, \ldots$$

where $a = \lambda/\mu$.

The symmetry property of the Charlier polynomials [6]

$$c_m(n; a) = c_n(m; a)$$

implies that

$$Q_m(n\mu) = Q_n(m\mu).$$

From (4), we therefore have the following explicit spectral representation of the transition probability:

$$\begin{aligned} P_{ij}(t) &= \frac{a^j}{j!}\sum_{r=0}^{\infty} \mathrm{e}^{-r\mu t} Q_i(r\mu) Q_j(r\mu) \mathrm{e}^{-a}\frac{a^r}{r!} \\ &= p(j)\sum_{r=0}^{\infty} \frac{\left(a\mathrm{e}^{-\mu t}\right)^r}{r!} c_i(r;a) c_j(r;a) \\ &= \frac{\mathrm{e}^{a(1-z)}}{j!}[a(1-z)]^j (1-z)^i c_i\left(j; -\frac{a(1-z)^2}{z}\right) \end{aligned} \tag{6}$$

where in arriving at (6) we have set $\mathrm{e}^{-\mu t} = z$ and made use of the following Meixner's bilinear generating formula for the Charlier polynomials [7]:

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} c_k(m;x) c_k(n;y) = \mathrm{e}^z \left(1 - \frac{z}{x}\right)^m \left(1 - \frac{z}{y}\right)^n c_m\left(n; -\frac{(x-z)(y-z)}{z}\right).$$

4. A multiple generating function

For the process $\{X(t), t \geqslant 0\}$, let us define

$$P_{s_1 s_1}(t_1) = \Pr[X(t_1) = s_1 | X(0) = s_1]$$

as the one-step recurrence probability at state s_1, and similarly

$$P_{s_1 s_1}(\tau_n) = \Pr[X(\tau_n) = s_1 | X(0) = s_1] \qquad \tau_n = \sum_{i=1}^{n} t_i$$

as the n-step recurrence probability. The sum

$$J_n = \sum_{s_1=0}^{\infty} P_{s_1 s_1}(\tau_n) \tag{7}$$

may thus be interpreted as the n-step recurrence probability for all the states in the time interval τ_n.

By virtue of the Markovian nature of the process, a repeated use of the Chapman–Kolmogorov equation in (5), for summations running from s_2 to s_n, also yields the following alternative expression for J_n:

$$J_n = \sum_{s_1, s_2, \ldots, s_n = 0}^{\infty} P_{s_1 s_2}(t_1) P_{s_2 s_3}(t_2) \cdots P_{s_{n-1} s_n}(t_{n-1}) P_{s_n s_1}(t_n). \tag{8}$$

We note from (6) that

$$P_{s_1 s_1}(\tau_n) = \mathrm{e}^{-a(1-\sigma_n)} {\sigma_n}^{s_1} L_{s_1}\left(-\frac{a(1-\sigma_n)^2}{\sigma_n}\right) \tag{9}$$

where as before, $\sigma_n = z_1 z_2 \cdots z_n$, $z_i = \mathrm{e}^{-\mu t_i}$ for $1 \leqslant i \leqslant n$, and note the fact [6] that

$$c_n(m;x) = (-x)^{-n} n! L_n^{(m-n)}(x). \tag{10}$$

Using the following well-known generating function for the simple Laguerre polynomials [8]:

$$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{1-t} \exp\left(-\frac{xt}{1-t}\right) \qquad |\, t \,| < 1$$

we obtain from (7) the explicit result

$$J_n = \frac{1}{1-\sigma_n}. \tag{11}$$

On the other hand, with (6) substituted for the various transition probabilities in (8), and using the connection between the Charlier and the generalized Laguerre polynomials from (10), we find

$$J_n = \exp -\{a(n-\sigma_1)\} \sum_{s_1, s_2, \ldots, s_n=0}^{\infty} Z_1^{s_1} Z_2^{s_2} \cdots Z_n^{s_n} L_{s_2}^{(s_1-s_2)}(X_1) L_{s_3}^{(s_2-s_3)}(X_2) \cdots L_{s_1}^{(s_n-s_1)}(X_n) \tag{12}$$

where we have defined

$$Z_i = \begin{cases} \dfrac{z_{i-1}(1-z_i)}{1-z_{i-1}} & 2 \leqslant i \leqslant n \\[2ex] \dfrac{z_n(1-z_1)}{1-z_n} & i = 1 \end{cases} \tag{13}$$

$$X_i = -a\frac{(1-z_i)^2}{z_i} \qquad 1 \leqslant i \leqslant n. \tag{14}$$

Equating the results in (11) and (12) we then have an expression for a multiple generating function for the product of generalized Laguerre polynomials of the same genre as that given in (1). Note however, that while the forms of the generating function from (11) and (12), and that in (1) are similar, the parameters of Z_i and X_i in the former case are not independent, their connection being expressed through the intrinsic variable $z_i = \exp -\mu t_i$ from the relations given in (13) and (14).

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